

## 4. Cooling Techniques

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### Liquid nitrogen and liquid helium

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1. Liquid nitrogen and liquid helium

2. Dilution cooling

3. Magnetic cooling

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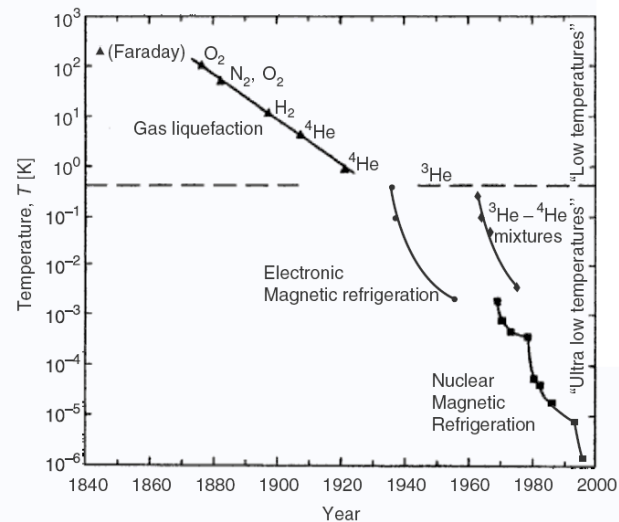
### Why low temperatures?

Many new physics can be observed at low temperatures. For the topics in these lectures, e.g.

- heat capacities of metals fall significantly below 100 K,
- Some metals can become superconducting at about 10 K,
- heat capacities for spin 1/2 salts peaks at around 1 K, and
- helium-3 becomes a superfluid below 2.5 mK.

We shall look at the main techniques for reaching these temperatures, and how statistical mechanics can help.

## The history of low temperatures



Pobell, Matter and methods at low temperatures (2007)

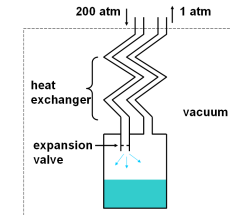
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## To liquefy gases

Lets start with the liquefying of gases. This is achieved by expansion.

In an ideal gas, expansion against a moving piston converts internal energy to work, thus cooling it down.



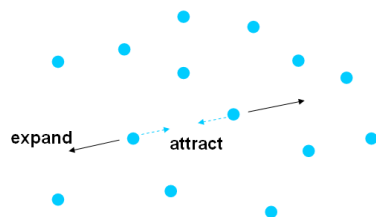
In a real gas, work must also be done against the attraction between atoms or molecules. This work is much larger than the work done against the moving piston, and is the principal means of cooling.

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## The Joule-Thomson (or Joule-Kelvin) effect

We know there is attraction between atoms or molecules in real gases.



When a real gas expands, molecules do work against this attraction as they move away from one another.

They lose kinetic energy and cools down. This is true for most gases, such as oxygen and nitrogen.

There are exceptions, like hydrogen and helium.

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## The Joule-Thomson (or Joule-Kelvin) effect

For hydrogen or helium, the atoms or molecules could hit each other so hard, that they experience very strong repulsion when they collide.



This increases their potential energy.

When allowed to expand and move further apart, this potential energy is converted to kinetic energy.

The gas would get hotter!

However, if the starting temperature is low enough, Helium can also cool when it expands.

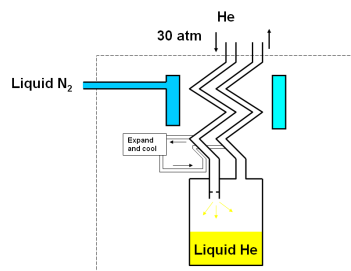
[http://en.wikipedia.org/wiki/Joule-Thomson\\_effect](http://en.wikipedia.org/wiki/Joule-Thomson_effect)

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## A helium liquefier

First, the helium gas is surrounded by liquid N<sub>2</sub> to "pre-cool" it.

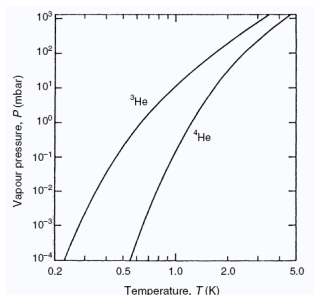


Then, 90% of it is expanded into another box for further cooling, and brought back in contact with the heat exchanger.

The rest expands into the bottom box. This is cold enough for a small fraction to become liquid helium (4.2 K).

## Liquid helium vapour pressure

Helium evaporation is an important cooling technique. The lowest temperature it can reach is limited by the vapour pressure,

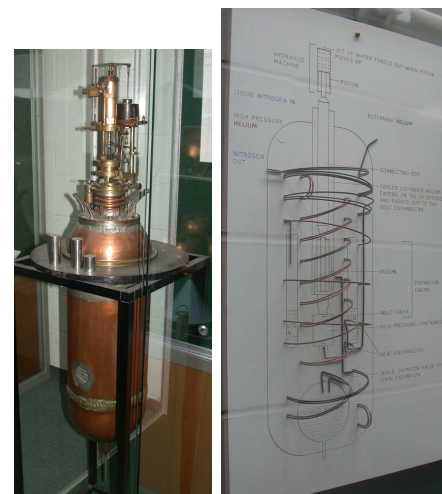


Pobell, Matter and methods at low temperatures (2007)

The graph shows that the vapour pressure  $P_{\text{vap}}$  falls exponentially with temperature,

$$P_{\text{vap}} \propto \exp(-1/T).$$

## Kapitza's helium liquefier

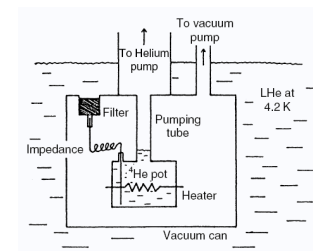


<http://www-outreach.phy.cam.ac.uk/camphy/museum/area7/cabinet1.htm>

## The helium-4 evaporation refrigerator

A liquid helium-4 bath on its own would have a temperature of about 4.2 K, the boiling temperature of helium-4.

It is possible to reduce this temperature by increasing the evaporation rate. This is done by pumping away the helium vapour.



Pobell, Matter and methods at low temperatures (2007)

Using this method, it is possible to reach 1.3 K. It is difficult to go below this because the vapour pressure decreases exponentially with falling temperature.

It should be mentioned that we can reach further down to 0.3 K if we use helium-3 instead of helium-4.

The main reason is that helium-3 has a higher vapour pressure, and therefore a higher vaporisation rate. This is due to the lower mass of the helium-3 atom.

The helium-3 evaporation refrigerator is similar in design to the helium-4 evaporation refrigerator.

The disadvantage of using helium-3 is the high cost.

Cooling by evaporation of helium-4 liquid can only reach about 1.3 K. Below this temperature, the vapour pressure is very small, so that very little would evaporate.

It is possible to overcome this limitation using a mixture of liquid helium-3 and liquid helium-4.

The way is to "evaporate" pure liquid helium-3 into the mixture.

This is done in the dilution refrigerator. Using this method, it is possible to reach into the milliKelvin range.

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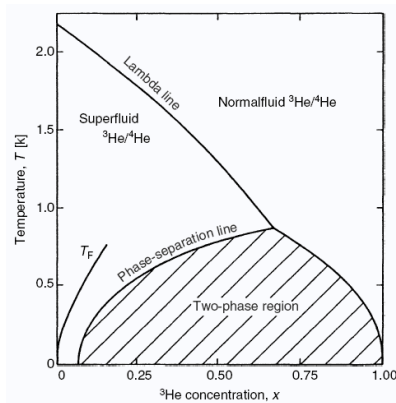
## Dilution Cooling

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## Properties of the liquid $^3\text{He}$ - $^4\text{He}$ mixtures

## Liquid $^3\text{He}$ - $^4\text{He}$ Mixtures

To understand the dilution refrigerator, we start with the phase diagram. This tells us how the mixture behaves at different concentrations and temperatures.



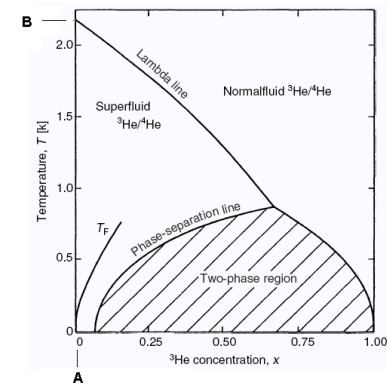
D.S. Betts: An Introduction to Millikelvin Technology (Cambridge University Press, 1989).  
J. Wilks, D.S. Betts: An Introduction to Liquid Helium, 2nd edn. (Clarendon, Oxford 1987)

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## The Phase Diagram

Let  $x$  be the fraction of  $^3\text{He}$  atoms in the  $^3\text{He}$ - $^4\text{He}$  mixture. In the figure,  $x$  is the horizontal axis.



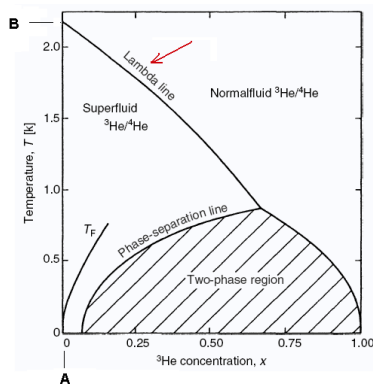
At point A:  $x = 0$  means pure  $^4\text{He}$ .

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## The Phase Diagram

The Lambda line indicates the temperature at which superfluid transition takes place for a given  $x$ .



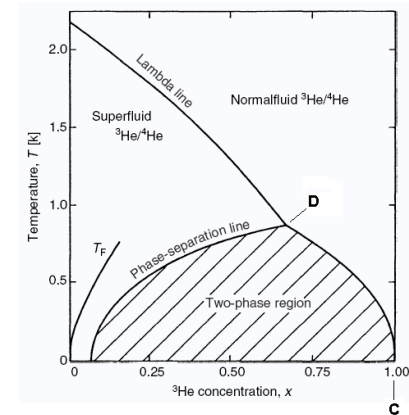
At point B: At  $x = 0$ , the Lambda line shows that superfluid transition takes place at the temperature  $T = 2.2$  K.

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## The Phase Diagram

At point C:  $x = 1$  is pure  $^3\text{He}$ .



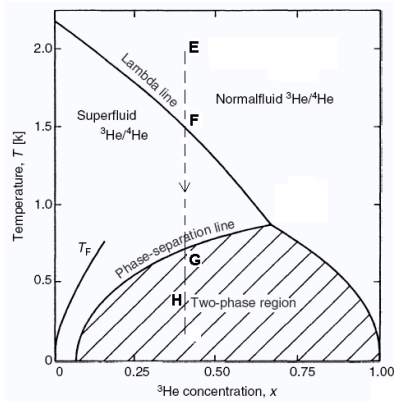
At point D: The Lambda line stops at the shaded region. So it is not clear if pure  $^3\text{He}$  (at C) can become superfluid. In fact it can, at the much lower temperature of 2.5 mK.

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## The Phase Diagram

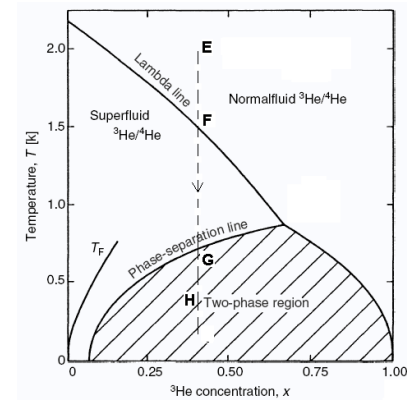
Along the dashed line: Suppose we start with a mixture at a concentration and temperature at point E. When this is cooled down to the temperature at point F, it would change to a superfluid.



If it is cooled further, it would reach the shaded region at G. What if it is cooled below that to a temperature at point H?

## The Phase Diagram

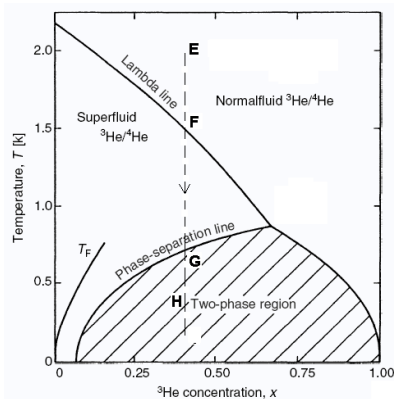
The shaded region is meant to indicate that it is not possible for the mixture to exist at a concentration and temperature in that region.



For a temperature at point H, it is only possible to have concentrations that are either smaller than that at point J, or higher than at point K.

## The Phase Diagram

So if a mixture is cooled down to a temperature at H, it would separate into two layers.

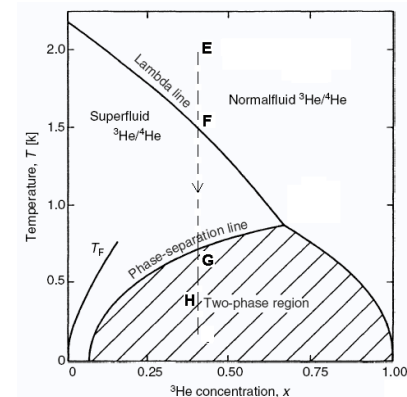


One layer has the concentration at J, and the other layer has the concentration at K.

## The Phase Diagram

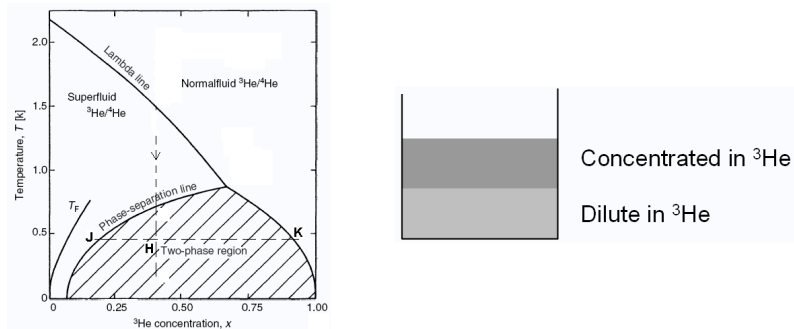
At J, the concentration  $x$  of  $^3\text{He}$  is lower, and the mixture is called the dilute phase.

At K, the concentration is higher, and it is called the concentrated phase.



## The Phase Diagram

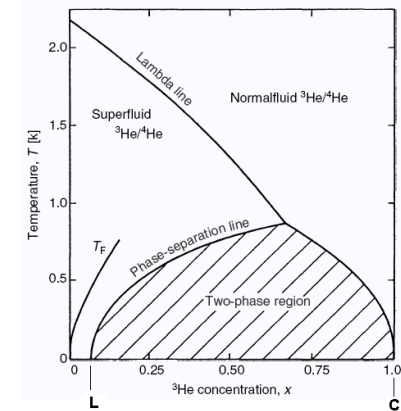
Because  $^3\text{He}$  is less dense than  $^4\text{He}$ , the concentrated phase floats on top of the dilute phase.



Note that the dilute phase is superfluid, whereas the concentrated phase is normal fluid.

## The Phase Diagram

At point L: When we reach a temperature below 0.1 K, the highest concentration possible is  $x = 6.6\%$ .

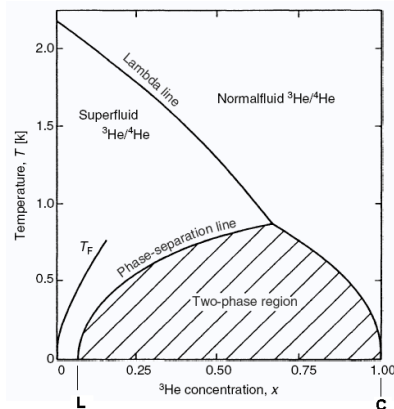


If we start at a higher temperature and concentration and cool down below 0.1 K, it would always separate into 2 layers:

1. One layer has 6.6% concentration in  $^3\text{He}$  - at L.
2. The other is nearly pure  $^3\text{He}$  - at C.

## The Phase Diagram

At point L: Note that in the dilute phase, the solubility of  $^3\text{He}$  would remain 6.6% even as the temperature approaches 0 K.



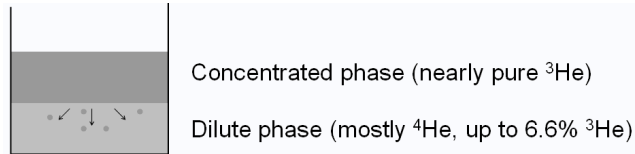
This is the property that makes the dilution refrigerator possible.

Cooling by dilution

## Cooling by dilution

Suppose that we start with a layer of pure helium-3 floating on a layer of pure helium-4 at 0.1 K.

Helium-3 would diffuse into the helium-4 layer below. The reverse would not happen, as we know from the phase diagram.



When this happens, the temperature would fall.

We may compare this to evaporation. Instead to vaporising to the vacuum above, the helium-3 "vaporises" into the liquid below.

In this case, it is mixing instead of evaporation.

## Cooling power

We have previously seen that the vapour pressure of helium falls exponentially with decreasing temperature.

The vapour pressure,  $P$ , is directly related to the rate at which the helium atoms vaporise from the liquid. Therefore it is also directly related to the rate of removing heat from the liquid:

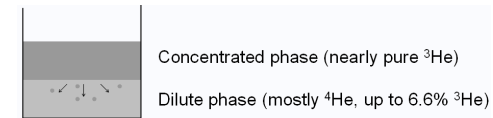
$$\dot{Q} \propto P \propto e^{-1/T}$$

$\dot{Q}$  is also called the cooling power, and is used to compare the performance of different refrigerators.

So the cooling power of the evaporation refrigerator falls exponentially with decreasing temperature.

## Cooling by dilution

By mixing into the lower layer, the helium-3 above is effectively being diluted. Hence the term "dilution cooling."



This continues, and the concentration of helium-3 in the in the bottom layer increases until it reaches 6.6%. Then the mixing stops.

In order to continue cooling, we must somehow remove the helium-3 dissolved in the dilute phase. This also has an analogy with cooling by evaporation, where we have to pump out the vapour to prevent it from being saturated.

## Cooling power

In dilution cooling, the helium-3 concentration,  $x$ , in the dilute phase is directly related to the rate at which helium-3 leaves the concentrated phase.

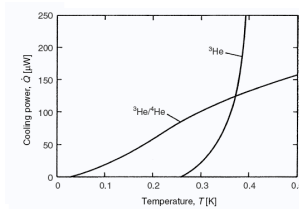
To determine the rate at which heat is removed, we must consider the heat change of mixing,  $\Delta H$ . As we shall see later, this heat change is proportional to  $T^2$ .

The cooling power of helium-3 dilution is therefore

$$\dot{Q} \propto x \Delta H \propto T^2$$



The figure compares the cooling power of helium-3 evaporation, and  $^3\text{He}$ - $^4\text{He}$  dilution, for the same helium-3 circulation rate.



assuming the same helium gas circulation rate of  $5\text{ l s}^{-1}$

O.V. Lounasmaa: Experimental Principles and Methods Below 1K (1974)

From previous reasonings, we can understand why the cooling power for the evaporation falls much faster than that for dilution.

Below 0.3 K, the dilution refrigerator clearly has a much higher cooling power than the evaporation refrigerator.

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 $^3\text{He}$ - $^4\text{He}$  mixtures as Fermi liquids

It is possible to derive the heat change of mixing. This would be useful for estimating the cooling power.

Helium-3 has nuclear spin  $I = 1/2$ . It is a fermion. It obeys the Fermi-Dirac statistics. So we hope that the same formula for the heat capacity of electrons can be used for the  $^3\text{He}$ - $^4\text{He}$  mixture:

$$C_3 = \frac{\pi^2}{2} \frac{T}{T_F} R \text{ at } T \ll T_F$$

where  $C_3$  denotes the heat capacity of helium-3.

From the lectures on electrons in metal, we have seen that the Fermi temperature  $T_F = E_F/k_B$ , and the Fermi energy is

$$E_F = \frac{\hbar^2}{2m_3} \left( \frac{3\pi^2 N}{V} \right)^{2/3}$$

where  $m_3$  denotes the mass of the helium-3 atom.

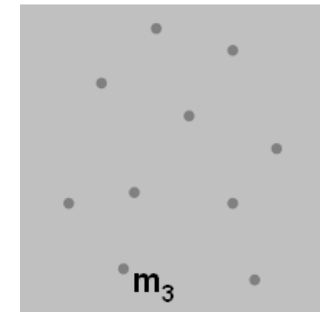
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 $^3\text{He}$ - $^4\text{He}$  mixtures as Fermi liquids

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 $^3\text{He}$ - $^4\text{He}$  mixtures as Fermi liquids

The problem with the formulae for electrons is that they are derived assuming that the particles do not interact with one another.



This assumption is not valid here. In the dilute phase, the helium-3 atoms are very close to the helium-4. In the concentrated phase, the helium-3 are very close to themselves.

It turns out that the helium-3 atoms in the dilute phase can be modelled as heavier particles in a vacuum.



For example, at 6.6% concentration, if we use a value of  $2.45m_3$  for the mass of helium-3 instead of the actual  $m_3$ , the formulae would still give a reasonable answer.

The higher mass is called the effective mass, and often denoted by  $m^*$ . This has been demonstrated experimentally in 1966.

A. C. Anderson, et al, Physical Review Letters, vol. 16 (1966), pp. 263-264

(For pure helium-3,  $m^* = 2.78m_3$ ).

### Heat change of mixing

Another point to note for the formula

$$C_3 = \frac{\pi^2 T}{2 T_F} R \text{ at}$$

is the condition that  $T \ll T_F$ .

For electrons in metal, we have seen that the Fermi energy is much higher than  $k_B T$  at room temperature.

What is the Fermi temperature for helium-3 in the dilute phase? Is it still higher than the temperature we are interested in?

At 6.6% concentration, and using the effective mass of  $m^* = 2.45m_3$ , we would find using the formulae that  $T_F$  is about 1 K.

The dilution refrigerator typically operates below 0.1 K. This should be well within the valid range for the Fermi gas formulae.

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### Heat change of mixing

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To derive the heat change of mixing, we need a few ideas and formulae from thermodynamics. Since the movement of particles from concentrated to dilute phase is essentially a change in phase, we need the condition for phase equilibrium:

$$\mu_C = \mu_D$$

where  $\mu$  is the chemical potential, subscript C is for concentrated phase, and D for the dilute phase.

The chemical potential is given by

$$\mu = H - TS$$

where  $H$  is the enthalpy per mole, and  $S$  the entropy per mole of the phase.

I shall start with a quick summary on the basic physics behind the equilibrium condition.

The enthalpy  $H$  is given by

$$H = U + pV,$$

where  $U$  is the internal energy,  $p$  the pressure and  $V$  the volume. So

$$\mu = H - TS = U + pV - TS.$$

The equilibrium equation is a statement that the change in chemical potential if one phase is changed to the other, is zero. This may also be expressed as:

$$\Delta\mu = \Delta U + p\Delta V - T\Delta S = 0.$$

This assumes that pressure and temperature are the same in both phases.

There would always be a pressure and a temperature gradient in the refrigerator, since the concentrated phase is being cooled and it is on top. But since the volumes are small and the two phases are in close contact, we shall assume that it is approximately true.

Returning to the equilibrium condition

$$\mu_C = \mu_D,$$

since  $\mu = H - TS$ , we may write this as

$$H_C - TS_C = H_D - TS_D$$

Remember that subscript C is for concentrated phase, and D for the dilute phase. The enthalpy change of mixing is therefore:

$$H_D - H_C = TS_D - TS_C$$

This is the heat change of mixing that we want to estimate.

To do so, we need to find the entropies  $S_C$  and  $S_D$  in both phases.

The equilibrium condition is:

$$\Delta U + p\Delta V - T\Delta S = 0.$$

To understand this physically, note that for a reversible change,  $T\Delta S = \Delta Q$ , the heat input.  $p\Delta V$  is the work done by one phase if it expands on changing to the other phase.

So the left hand side is a statement that the total energy of one mole of a phase remains the same, when it changes into another phase.

If this is the case, then the two phases would remain in equilibrium.

If, on the other hand, there is a net release in energy when one phase changes into the other, then this change would take place, and there would be no equilibrium.

What we need the entropy in each phase. This can be determined from the specific heat capacity as follows.

Conservation of energy, or the first law of thermodynamics, tells us that:

$$dQ = dU + dW$$

$dW$  is the mechanical work done - by expansion or contraction. Since we have a liquid, the volume change is very small. If we neglect this, we have

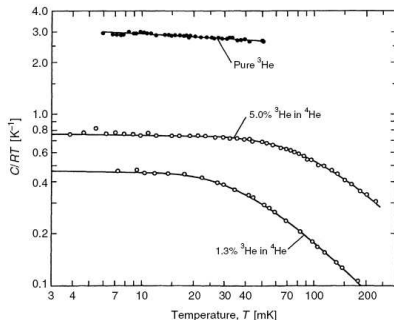
$$dQ = dU$$

Since  $dS = dQ/T$ , the entropy is given by integrating:

$$S = \int \frac{dQ}{T} = \int \frac{dU}{T} = \int \frac{C}{T} dT$$

In order to obtain the entropy, we need know the specific heat capacity as a function of temperature.

We start with the concentrated phase, which we shall treat as pure helium-3.



J.C. Wheatley: Am. J. Phys. 36, 181 (1968)  
A.C. Anderson, et al: Phys. Rev. Lett. 16, 263 (1966)

The graph above shows the experimental measurements on the specific heat for pure helium-3 on top, and two mixtures below.

In the dilute phase, the specific heat for helium-3 may be estimated by the Fermi gas formula

$$C_3 = \frac{\pi^2}{2} \frac{T}{T_F} R$$

Suppose that the concentration of the mixture in a dilution refrigerator is close to 6.6%. Using the effective mass of  $m^* = 2.45m_3$ , and the formula for the Fermi energy, we would get

$$C_3 = 106T \text{ J/(mol K)}$$

So the entropy for helium-3 in the dilute phase is

$$S_D = \int_0^T \frac{C_3(T')}{T'} dT' = \int_0^T 106 dT' = 106T \text{ J/(mol K)}$$

By fitting the data, we obtain for pure helium-3 an approximate formula for the specific heat:

$$C_3 = 22T \text{ J/(mol K)}$$

Note the graph only shows measurement up to 40 mK, so this formula is only valid below that.

The entropy for the concentrated phase is therefore

$$S_C = \int_0^T \frac{C_3(T')}{T'} dT' = \int_0^T 22 dT' = 22T \text{ J/(mol K)}$$

In principle, we can use the same method to find the entropy for helium-3 in the dilute phase. Unfortunately, we would need this for different concentrations, and there are not many measurements available.

We have seen earlier that the specific heat formulae for a Fermi gas can be used ...

We have obtained the entropies in the two phases:

$$S_C = 22T \text{ J/(mol K)}$$

$$S_D = 106T \text{ J/(mol K)}$$

We can now find the heat change of mixing:

$$H_D - H_C = T(S_D - S_C) = T(106T - 22T) = 84T^2 \text{ J/mol}$$

Note that this change is positive. This implies an increase in internal energy.

Because of conservation of energy, heat energy has to be absorbed from the surrounding. Hence the cooling.

The cooling power is then

$$\dot{Q} = \dot{n}_3 \Delta H = \dot{n}_3 (H_D - H_C) = 84 \dot{n}_3 T^2 \text{ W}$$

where  $\dot{n}_3$  is the number of moles per second of helium-3 moving from concentrated to dilute phase.

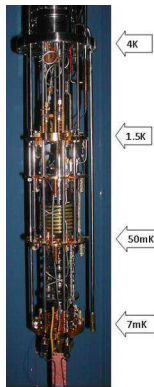
The cooling power is proportional to  $T^2$ , as we have seen early on.

For typical values of  $\dot{n}_3 = 100 \mu\text{mol s}^{-1}$  and  $T = 10 \text{ mK}$ , we find

$$\dot{Q} = 1 \mu\text{W}.$$

### Examples of dilution refrigerators

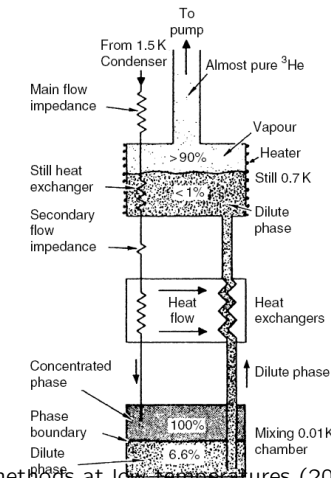
This is a dilution refrigerator used in the semiconductor physics group in the Cambridge physics department.



<http://www.phy.cam.ac.uk/research/sp/cryo.php>

You should be able to tell from the temperatures indicated what the various parts are.

This is a schematic diagram. It would be housed inside a vacuum chamber that is immersed in a helium-4 bath at 4.2 K.



Pobell, Matter and methods at low temperatures (2007)

## Magnetic Cooling

We have seen that using dilution of helium-3, we could reach temperatures in the milliKelvin range.

If we want to go below that, a completely new technology is needed. This can be achieved using magnetic cooling.

Historically, magnetic cooling has developed in 2 stages.

When it was first proposed in the 1920s, paramagnetic salts was used for cooling. Today, this method can reach down to milliKelvin temperatures. The use of paramagnetic salt has now been largely replaced by the dilution refrigerator.

In the 1950s, the use of the magnetic moment of nuclei in metals was started. This method can now reach microKelvin temperatures.

Whether it is magnetic moments of electrons in salts or nuclei in metals, the principle is the same.

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### The Principle of Magnetic Cooling

Consider a paramagnetic salt. This salt contains ions with magnetic moments coming from their electrons.

At mK temperatures, the magnetic disorder entropy (about 1 J/mol) is large compared to all other entropies, such as lattice and conduction electron entropies, which may be neglected.

We have previously looked at the properties of a paramagnetic salt. The entropy would be a function of the magnetic field applied. We shall look at the derivation of the entropy formula later.

First, we look at the known behaviour of the entropy and see how this can help us to understand magnetic cooling.

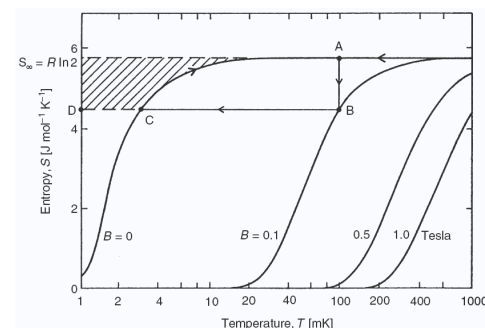
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### The Principle of Magnetic Cooling

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### The Principle of Magnetic Cooling

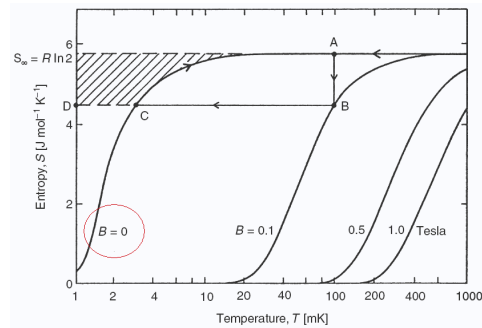
This graph shows the entropy against temperature for a commonly used paramagnetic salt.



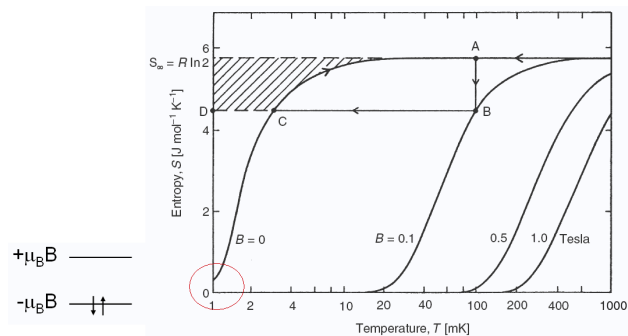
S.A.J. Wieggers, P.E. Wolf, L. Puech: Physica B 165 & 166, 165 (1990)

We start by familiarising ourselves with the various features of the graph.

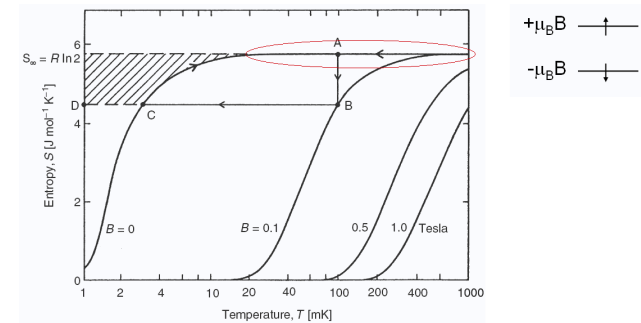
The symbol  $B$  is the magnetic field. We start with the graph for zero or low magnetic field.



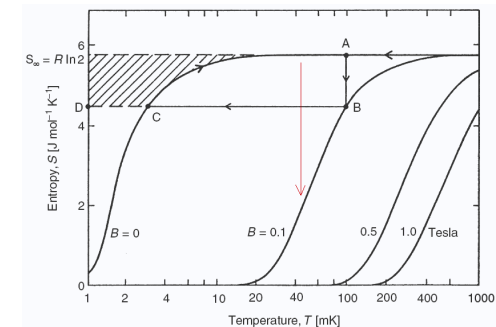
At low temperature, the entropy goes to zero because all particles fall to the lowest level.



At high temperature, the entropy approaches a constant because it becomes equally likely to be at any of the magnetic energy levels.



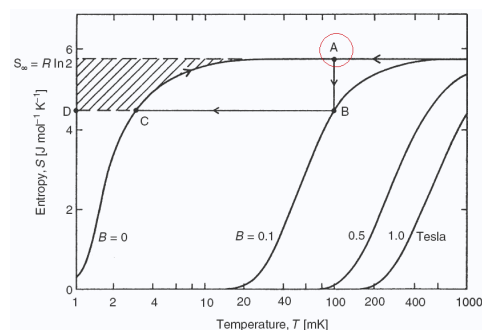
Next, suppose the magnetic field is increased, say from 0 T to 0.1 T. The spacing between energy level would increase.



Then it becomes more likely for a particle to be at the lower level. So the entropy would fall.

## The Principle of Magnetic Cooling

Point A: To understand how to use the magnetic property for cooling, suppose we start with a temperature at point A, and with a low magnetic field..

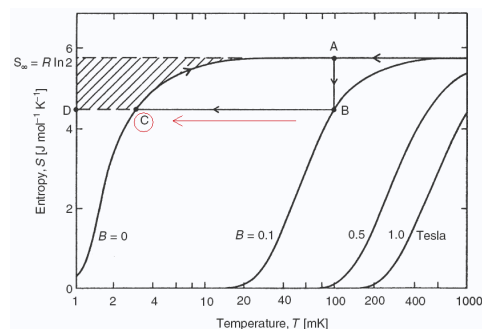


The salt is placed in contact with a precooling bath. This can either be a helium bath, or a dilution refrigerator.

## The Principle of Magnetic Cooling

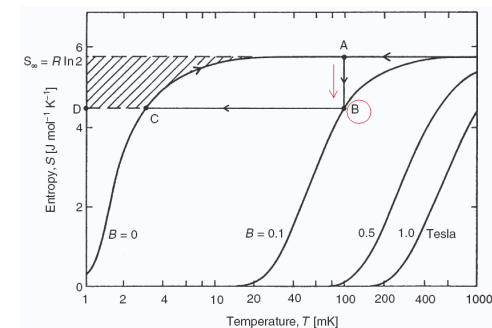
The salt is then thermally isolated from the precooling bath (e.g. by using a heat switch).

Point C: Demagnetisation now takes place adiabatically (so entropy is constant). The magnetic field is reduced to a very small value. The temperature falls to C.



## The Principle of Magnetic Cooling

Point B: A magnetic field is then applied. This is done isothermally. The entropy falls to point B at constant temperature.



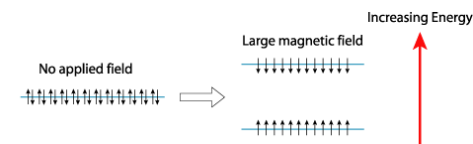
This process performs magnetic "work" on the salt, which is converted to heat (like compressing a gas). This heat would be absorbed by the precooling bath.

## How does it work?

We have seen how the cooling takes place using thermodynamics. Let us now see how this takes place physically.

The ions in the salt have magnetic dipole moments. Normally, half of the dipoles are spin up, and the other half are spin down.

If a strong magnetic field is applied, the energy levels will split into two.

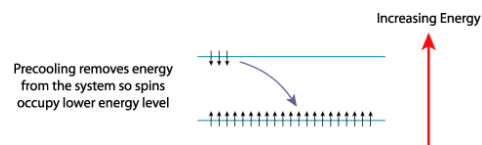


Dipoles in the direction of the field will have lower energy, and dipoles in the opposite direction higher energy.



Remember that a helium bath or a dilution refrigerator is cooling the salt at the same time.

This would remove energy from the higher energy atoms, so that they fall into the lower energy state.

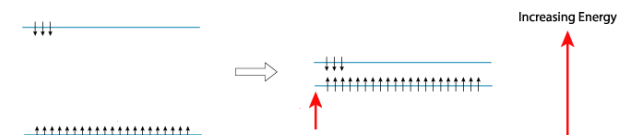


This is the "precooling."

Then, using a heat switch, the salt can be thermally disconnected from the precooling bath.

The magnetic field is now slowly reduced.

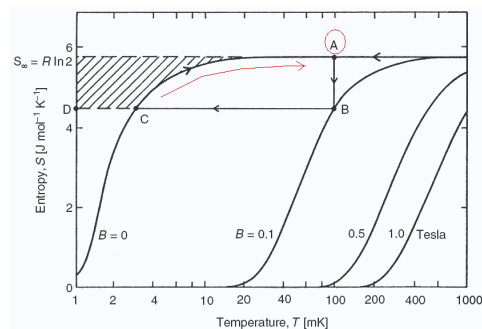
The lower energy level, which contains most of the atoms, is then forced to increase in energy.



This energy has to come from the surrounding. So the salt cools down.

### The Principle of Magnetic Cooling

Point C: After some time, the salt would warm up because of heat leak from the surroundings, since the insulation is not perfect. The temperature returns to point A along the curve from C.



The temperature cannot be maintained, unlike the dilution refrigerator. This is called a "one-shot" technique.

### Thermodynamics of Magnetic Cooling

With the help of the entropy graph, we can calculate the heat  $Q$  and temperature  $T$  in magnetic cooling.

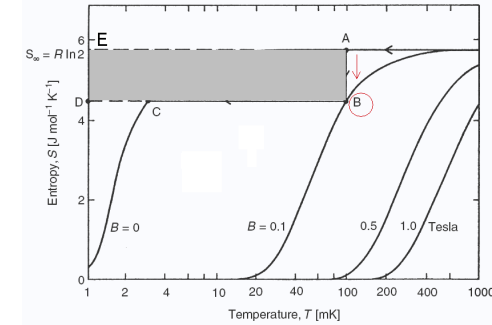
We can divide the cycle into the following stages:

1. Isothermal magnetisation: We shall find the heat given out by the salt.
2. Adiabatic demagnetisation: We shall find the lowest temperature reached.
3. Warming up: We shall determine the cooling power.

Isothermal magnetisation takes place from A to B on the graph at the start. Since  $dQ = TdS$ , the heat given out is

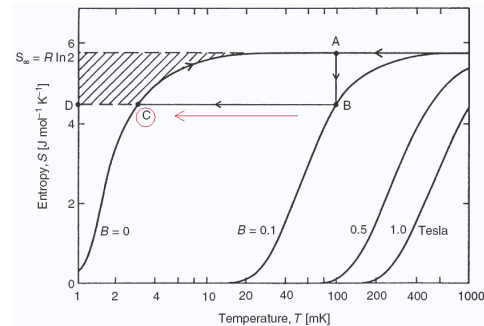
$$Q = \int_B^A TdS.$$

This is just the area of the rectangle ABDE.



The heat released is usually a few J/mol of the refrigerant (the salt), so it can easily be absorbed by an evaporating helium bath or a dilution refrigerator.

Adiabatic demagnetisation takes place from B to C on the graph.



Later on, we shall derive the formula for magnetic entropy. In that formula, we shall see that the entropy is a function of  $B/T$ .

For now, a quick way to understand this is to look at the Boltzmann distribution,  $\exp(-\mu_B B/k_B T)$ . This is indeed a function of  $B/T$ .

Since the entropy should depend on  $\exp(-\mu_B B/k_B T)$ , we would expect the entropy to be a function of  $B/T$  as well.

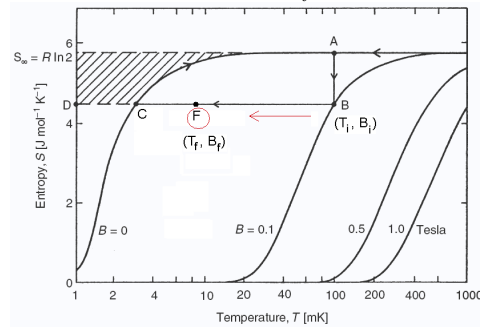
This gives us a quick way to find the coldest temperature reached in this demagnetisation step.

From the graph, we see that the entropy is a simple function of  $T$  and  $B$ . If  $T$  increases, entropy increases. If  $B$  increases, entropy decreases.

In the adiabatic process, the entropy is a constant. Since it is a function of  $B/T$ , then  $B/T$  must also be constant. This is just the equation we need:

$$\frac{B}{T} = \text{constant}$$

Point B: Suppose we start at temperature  $T_i$  and field  $B_i$ . If this is point B on the graph, then  $B_i = 0.1T$  and  $T_i = 100mK$



Point F: We then reduce the field to a smaller value  $B_f$ . Let  $T_f$  be the new temperature.

The equation  $B/T = \text{constant}$  means that

$$\frac{B_f}{T_f} = \frac{B_i}{T_i}$$

### Interacting Magnetic Dipoles

The new temperature is

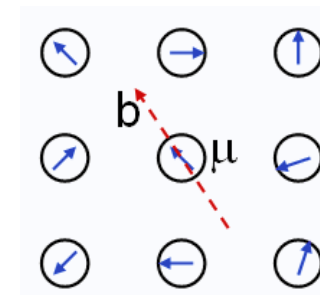
$$T_f = \frac{T_i}{B_i} B_f$$

Clearly, we can make  $T_f$  very small by reducing the magnetic field  $B_f$  to a very small value.

But what if we reduce the magnetic field  $B_f$  to zero? Surely the temperature  $T_f$  would not go to zero. Something must happen to limit the lowest temperature that we can reach.

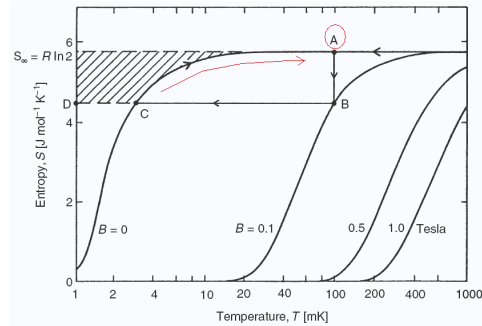
Indeed, this is the case. When the temperature is sufficiently low, the effects of the magnetic fields from neighbouring ions of the salt become important.

When the temperature is sufficiently low, the mutual interaction would tend to align all magnetic dipoles in the same direction. When this happens, the entropy falls to zero, and the magnetic cooling would stop.



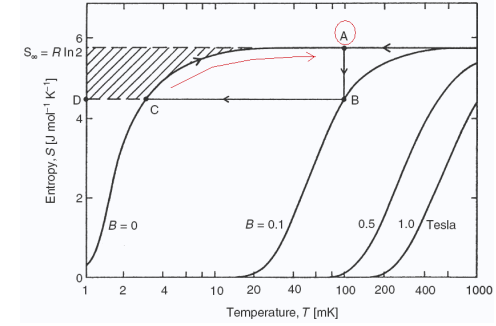
So the mutual interaction limits the lowest temperature that can be achieved using this method. At very low temperatures, the equation would have to be modified to take into account this mutual interaction. We shall come back to this.

Point C: We can now see that the graph is misleading. We do not actually demagnetise to a field at point C, which is zero. Rather, we would demagnetise to a very small field, close to C.



To make things simple, we shall still refer to C for the end point of the demagnetisation. Then warming up starts.

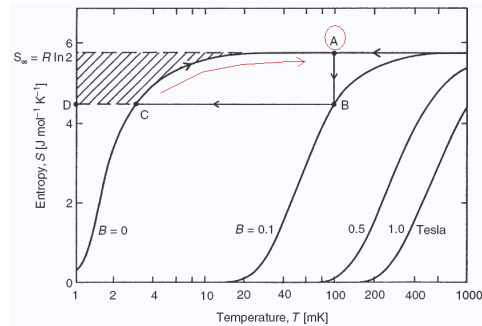
Remember that the salt is thermally isolated during the demagnetisation. After reaching the lowest temperature at C, the salt remains isolated. We hope that it would stay cold for as long as possible.



But because insulation is not perfect, the salt starts warming up slowly. Since the magnetic field  $B$  is fixed, the temperature and entropy would follow the curve and eventually reach the starting temperature at A.

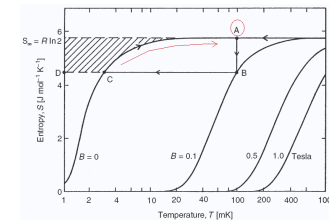
As it warms up, the heat absorbed by the salt can be obtained from the entropy using the same formula as before

$$Q = \int_C^A T dS$$



We need to integrate along the curve from C to A. So the heat absorbed is given by the shaded region on the graph.

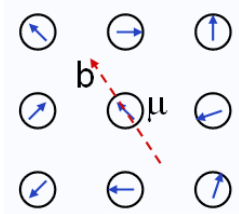
The heat absorbed in warming up also gives the cooling power. If the salt can absorb more of the heat that leaks in through the insulation, then it would be able to remain cold for a longer period of time.



Note that this is a different definition from before. For the dilution refrigerator, the cooling power  $\dot{Q}$  is the rate at which heat is absorbed.

**The cooling power  $Q$  for the magnetic refrigerator is the total heat absorbed.**

We have seen that the interaction between magnetic dipoles of the ions in the salt sets a lower limit to the temperature that can be reached by demagnetisation.



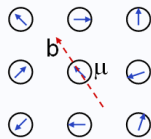
This also means that the formula for the lowest temperature

$$T_f = \frac{T_i}{B_i} B_f$$

would not be accurate at very low temperatures. It can be modified to the following form:

$$T_f = \frac{T_i}{B_i} \sqrt{B_f^2 + b^2}$$

When the applied magnetic field is reduced to zero, there is indeed a remaining field. That would be the resultant field from the neighbouring magnetic dipoles.



We are looking at a temperature  $T_c$  at which the effect of this interaction becomes important. This means that  $k_B T_c$  is comparable to the interaction energy

$$\varepsilon_d = \mu b.$$

$T_c$  is called the ordering temperature. It is the temperature below which the neighbouring fields become strong enough to align the dipoles. We may define

$$k_B T_c = \mu b.$$

Lets try and understand this formula physically.

$$T_f = \frac{T_i}{B_i} \sqrt{B_f^2 + b^2}$$

We see that when  $B_f$  is reduced to zero,

$$T_f = \frac{T_i}{B_i} b.$$

If we compare this with the original form of

$$T_f = \frac{T_i}{B_i} B_f$$

we see that  $b$  corresponds to  $B_f$ . This makes sense if we think of  $b$  as the field that remains after the applied magnetic field has been reduced to zero.

## 6.5 Magnetic Refrigerators

The performance of a magnetic refrigerator is mainly determined by:

- the starting magnetic field and temperature,
- the heat leaks, and
- the paramagnetic salt that is used.

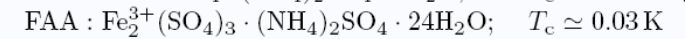
Typical starting conditions are 0.1 to 1 T and 0.1 to 1 K. These are fairly easy to achieve nowadays.

There are a few properties of a paramagnetic salt that are desirable:

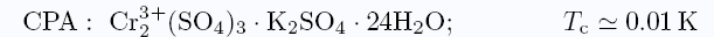
- low ordering temperature to reach low temperatures,
- large specific heat to absorb more heat before warming up

The following are paramagnetic salts that have been used:

“High”-temperature salts:

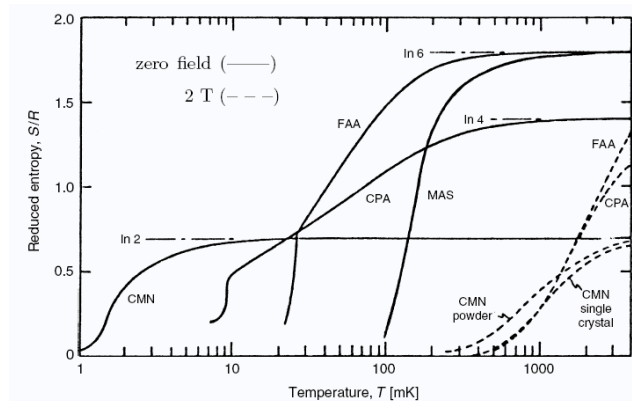


“Low”-temperature salts:



The last one, CMN, has the lowest ordering temperature. This means it can potentially reach the lowest temperature before the interaction between magnetic dipoles become important. CMN has been used extensively and could reach 2 mK.

The following graphs show the entropies of the actual salts.

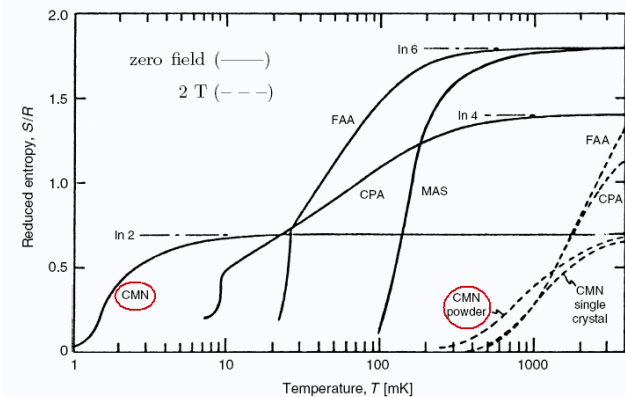


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For example, look at the 2 graphs for the salt CMN:

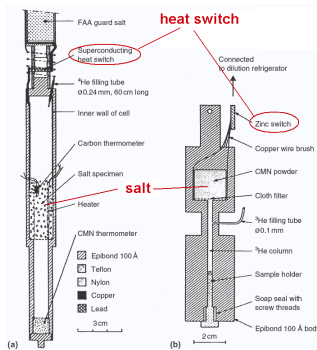
The one to the left is for zero magnetic field.

The one to the right is for 2 T.



They look very similar to the entropy graphs that are shown earlier.

These are examples of actual magnetic refrigerators that have been built.



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Notice the heat switch near the top. They are usually connected to a dilution refrigerator. The paramagnetic salt refrigerant is in the middle.

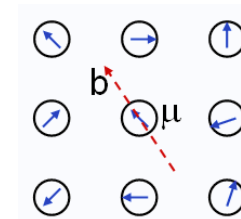
## 6.6 Nuclear Refrigeration

Magnetic refrigerators using paramagnetic salts are now largely replaced by the dilution refrigerator, which can reach the same temperatures.

However, they are still useful for small experiments and satellites, where compact refrigerators are required. Examples are in detectors for millimetre wave, X rays and dark matter.

## Nuclear Refrigeration

We have so far looked at the use of the electronic magnetic dipoles for cooling. This is limited to milliKelvin temperatures by the interaction between the electronic dipoles.



It is possible to reach much lower temperatures if we use the nuclear magnetic dipoles. The magnetic dipole moment of the nucleus is much smaller than that of the electron. As a result, the interaction between nuclear dipoles is much weaker.

To get some idea of the relative magnitudes, we look at the unit for electronic dipole moment (Bohr magneton) and the unit for nuclear dipole moment (nuclear magneton):

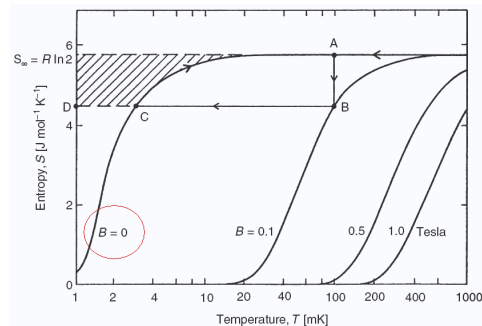
Bohr magneton,  $\mu_B = 9.27 \times 10^{-24} \text{ J/T}$   
 Nuclear magneton,  $\mu_n = 5.05 \times 10^{-27} \text{ J/T}$

The nuclear magneton is nearly 2000 times smaller. This gives us an idea of how much smaller the nuclear magnetic moment is.

If we use the nuclear magnetic dipole for cooling, we can reach microKelvin temperatures because of the much smaller interaction field. The ordering temperature for the nuclear dipole can be as small as  $0.1 \mu\text{K}$ .

For nuclear cooling, we can use metal as the refrigerant instead of paramagnetic salts. Metal has the advantage of high thermal conductivity.

From the entropy graph, we can see that to reduce the entropy further, even higher fields and lower temperatures would be required.



The 8 T field already requires superconducting magnets, and the 10 mK temperature would require a dilution refrigerator.

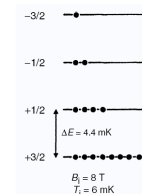
Although the small nuclear moment offers the potential of reaching much lower temperatures, it also requires much more demanding conditions. The very small moment means that we need very high starting magnetic fields, and very low starting temperatures.

As an example, we look at copper. Copper is a "work horse" of nuclear refrigeration. For copper, we would typically need a starting field of  $B_i = 8 \text{ T}$ , and a starting temperature of  $T_i = 10 \text{ mK}$ . This is just to reduce the entropy by 9%.

From the earlier explanation on the principle of magnetic cooling, we know that a lower entropy means that a lower temperature can be reached during demagnetisation. It also means a higher cooling power, since more heat can be absorbed during warming up.

Why would a small magnetic moment require higher starting field and lower starting temperature? Let's try and understand this physically.

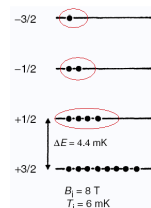
Consider the magnetic energy levels of copper. Applying a magnetic field increases the spacing between levels. Because of the small nuclear moment, the spacing would be small even for a high starting field.



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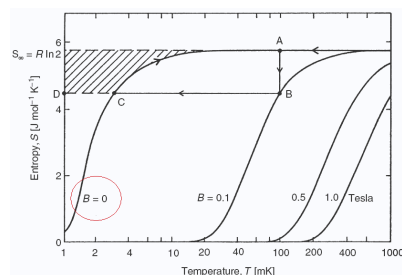
Because the spacing is small, we would get more particles at the higher energy level according to the Boltzmann distribution. This means higher entropy. In order to reduce this, we need a lower starting temperature.



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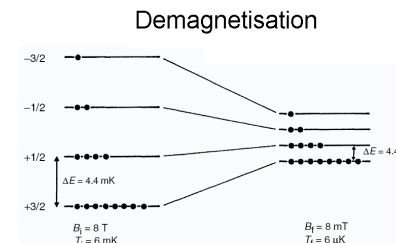
This is difficult. For copper, even at a starting temperature of 10 mK, there is still a substantial fraction of the nuclei at the higher levels. The levels are just too close together because of the small nuclear moment.

One disadvantage of very low temperatures in magnetic cooling is that the cooling power becomes very small. The cooling power is given by the shaded region in the graph.



The horizontal axis is temperature. So for low temperatures, the horizontal size of the shaded area would also be small. Since nuclear cooling is 1000 times colder than electronic magnetic cooling, the cooling power is also 1000 times smaller.

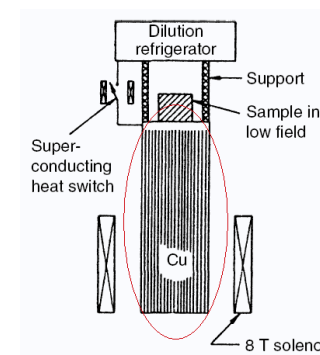
Fortunately, because the nuclear moment is small, the interaction field is also small. This means that in the demagnetisation step, it is possible to reduce the temperature to a very small value.



In the case of the copper example, if we reduce the field by 1000 times to 8 mT, the temperature also falls by 1000 times to 6 μK.

This is a schematic diagram of a nuclear refrigerator. Notice how the main components are connected together.

The refrigerant is a copper block at the centre.



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## Exercises

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### Answer

The units of the heat capacity is given as J/(mol.K). This refers for 1 mole of the liquid.

So the heat capacity formula for insulators at low temperature can be written as

$$C = \frac{12}{5} \pi^4 N_A k_B \left( \frac{T}{\theta_D} \right)^3.$$

where the number of particles  $N$  is replaced by Avogadro's constant  $N_A$ .

The power of 3 agrees with the gradient of 3 from the log-log plot.

Substituting the given data:

$$T = 0.5 \text{ K}, C_V = 0.01 \text{ J/(mol.K)}$$

into the above formula, we can then solve for  $\theta_D$ . The answer is 29.0 K.

---

## Exercise 1

Calculate the Debye temperature corresponding to the heat capacity  $C_V$  of liquid helium-4 for temperature  $T < 0.5 \text{ K}$ . The following data is obtained from a graph the measured data in which  $\log C_V$  is plotted against  $\log T$ .

log-log plot of the measured data:

$$T = 0.5 \text{ K}, C_V = 0.01 \text{ J/(mol.K)} \\ \text{gradient} = 3$$

---

## Exercise 2

At which temperature do the lattice and conduction electron contributions do the heat capacity of copper become equal?

The following data are obtained from a straight line graph of  $C/T$  plotted against  $T^2$ , where  $C$  is the measured heat capacity and  $T$  is the temperature:

$$\text{gradient} = 0.0469 \text{ mJ mol}^{-1} \text{ K}^{-4} \\ \text{vertical intercept} = 0.7 \text{ mJ mol}^{-1} \text{ K}^{-2}$$

Since the graph of  $C/T$  against  $T^2$  is a straight line, they are related by the straight line equation:

$$\frac{C}{T} = \gamma + AT^2.$$

Multiplying by  $T$ , we get

$$C = \gamma T + AT^3.$$

The electronic contribution is linear in  $T$ , so it would be given by the first term:

$$C_e = \gamma T.$$

The lattice (phonon) contribution is proportional to  $T^3$ , so it would be the second term:

$$C_{ph} = AT^3.$$

When they become equal, we can solve these 2 equations for  $T$ . This gives:

$$T = \sqrt{\frac{\gamma}{A}}.$$

### Exercise 3

At which temperature is the nuclear magnetic contribution in a field of 100 mT equal to the conduction electron contribution to the heat capacity of silver?

[You are given that the nuclear spin of silver is 1/2. The nuclear magneton is  $\mu_n = 5.051 \times 10^{-27} \text{ J T}^{-1}$ . For the heat capacity of silver,  $\gamma = 0.640 \text{ mJ mol}^{-1} \text{ K}^{-2}$ .]

We can find  $\gamma$  and  $A$  from the graph. Returning to the straight line equation

$$\frac{C}{T} = \gamma + AT^2.$$

we can see that  $\gamma$  would be the vertical intercept, and  $A$  would be the gradient. These 2 values are given. Substituting, we find:

$$T = \sqrt{\frac{0.7}{0.0469}} = 3.86 \text{ K}.$$

Remark: Since the phonon contribution varies as  $T^3$ , it would fall off much faster than the electronic contribution as temperature decreases. So below 3.86 K, the electrons would be the main contributor, and the heat capacity would be approximately linear.

On the higher temperature side of peak of the magnetic contribution, the magnetic heat capacity per mole is given by

$$C_m \rightarrow N_A k_B \left( \frac{\Delta E}{2k_B T} \right)^2$$

where

$$\Delta E = 2\mu_n B.$$

At low temperature, the electronic heat capacity per mole of copper is

$$C_e = \gamma T.$$

When the electronic and nuclear magnetic contributions become equal,

$$C_e = C_m.$$

Solving for  $T$ , we get

$$T^3 = \frac{N_A k_B}{\gamma} \left( \frac{\Delta E}{2k_B} \right)^2.$$

---

Substituting the values given and the constants, we find 25.9 mK.

Remark: So below this temperature, we would find that the heat capacity stops falling and rise instead. It would then be dominated by the magnetic contribution.

---

Answer

---

The Fermi temperature is given by

$$T_F = \frac{E_F}{k_B},$$

where the Fermi energy is

$$E_F = \frac{\hbar^2}{2m} \left( \frac{3\pi^2 N}{V} \right)^{2/3}.$$

Using the molar volume  $V_m$  that is given,  $N$  would be the Avogadro's constant  $N_A$ .

The mass of the helium-3 atom is  $3u$ , and the mass  $m$  in the formula for the Fermi energy would be the effective mass

$$m^* = 2.78m_3 = 2.78 \times 3u,$$

where  $u$  is the atomic mass unit.

---

**Exercise 4**

Calculate the Fermi temperature of liquid helium-3.

[You are given the following:

The volume  $V_m$  of 1 mole of liquid helium-3 is  $36.84 \text{ cm}^3$ .

The effective mass is  $m^* = 2.78m_3$ ,

where  $m_3$  is the mass of the helium-3 atom.

The atomic mass unit  $u$  is  $1.6605 \times 10^{-27} \text{ kg}$ .]

Substituting all these values into the Fermi energy equation, we find

$$E_F = 5.31 \times 10^{-23} \text{ J}.$$

Then the Fermi temperature is

$$T_F = \frac{E_F}{k_B} = 1.79 \text{ K}.$$

---

### Exercise 5

At which temperature would the Fermi heat capacity of liquid helium-3 reach the classical ideal gas value

$$C_P = \frac{5}{2}R$$

if its heat capacity continued to vary with temperature, as expressed by the Fermi gas formula

$$C_3 = \frac{\pi^2}{2} \frac{T}{T_F} R.$$

[You are given the following:

The Fermi temperature  $T_F$  of liquid helium 3 is 1.79 K.

The ideal gas constant  $R = 8.315 \text{ J}/(\text{mol K})$ .]

---

### Exercise 6

Assuming that liquid helium-3 is a Fermi gas, show that the entropy of liquid helium-3 is given by

$$S = C_3$$

where  $C_3$  is the heat capacity. Show that the enthalpy change is given by

$$\Delta H = T(C_D - C_C)$$

where  $C_D$  and  $C_C$  are the heat capacities in the dilute phase and the concentrated phase respectively.

---

### Answer

Substituting the given values of  $T_F$  and  $R$  into the Fermi gas formula

$$C_3 = \frac{\pi^2}{2} \frac{T}{T_F} R,$$

we get

$$C_3 = 22.9T \text{ [J}/(\text{mol.K})].$$

This would become equal to the classical value

$$C_P = \frac{5}{2}R$$

when

$$22.9T = \frac{5}{2}R.$$

Solving for  $T$  gives 0.91 K.

Remark: Notice that this is smaller than the Fermi temperature of 1.79 K. So for the helium-3 to behave like a Fermi gas, the temperature must be much smaller than this value.

---

### Answer

The Fermi gas heat capacity is given by

$$C_3 = \frac{\pi^2}{2} \frac{T}{T_F} R.$$

The entropy is

$$S = \int \frac{dQ}{T} = \int \frac{C_3 dT}{T}.$$

Substituting, we get

$$S = \int \frac{\pi^2}{2} \frac{1}{T_F} R dT = \frac{\pi^2}{2} \frac{T}{T_F} R$$

which is the same as  $C_3$ .

---

At phase equilibrium, the condition is

$$H_C - TS_C = H_D - TS_D$$

so the enthalpy change is

$$\Delta H = H_D - H_C = T(S_D - S_C).$$

Since the entropy is equal to the heat capacity, we have

$$\Delta H = T(C_D - C_C).$$

---

Answer

The heat capacity is inversely proportional to the Fermi temperature, as we can see from its formula:

$$C = \frac{\pi^2 T}{2 T_F} R.$$

The Fermi temperature  $T_F$  is directly related to the Fermi energy  $E_F$ , which is

$$E_F = \frac{\hbar^2}{2m} \left( \frac{3\pi^2 N}{V} \right)^{2/3}.$$

This is in turn inversely proportional to the mass  $m$ . It is also proportional to  $(N/V)^{2/3}$ , where  $N/V$  is the molar concentration.

We are given that  $r$  is the effective mass ratio, and  $x$  is the fractional concentration. So

$$E_F \propto \frac{x^{2/3}}{r}$$

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**Exercise 7**

The heat capacity of a Fermi gas is given by

$$C = \frac{\pi^2 T}{2 T_F} R.$$

Assuming that a mixture of helium-3 in liquid helium-4 behaves as a Fermi gas, show that

$$C \propto \frac{r}{x^{2/3}}$$

where  $r$  is the effective mass ratio, and  $x$  is the fractional concentration in the dilute phase.

We have found that

$$E_F \propto \frac{x^{2/3}}{r}$$

Therefore, since the heat capacity  $C$  is inversely proportional to the Fermi temperature  $T_F$ , we have

$$C \propto \frac{r}{x^{2/3}}.$$

Remark: This tells us the heat capacity of helium-3 is higher when the concentration is lower, and that it is proportional to the effective mass.

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### Exercise 8

The effective mass ratio of pure helium-3 increases from about 2.8 at saturated vapour pressure to about 4.6 at 20 bar. In the same pressure range, the effective mass ratio for a saturated mixture increases from about 2.4 to 2.8. How would the numerical value for the cooling power of a dilution refrigerator given by

$$\dot{Q} = 84\dot{n}_3 T^2$$

change if this refrigerator were operated at 20 bar.

[You are given that the heat capacity of helium-3 is related to the effective mass ratio  $r$ , and the fractional concentration  $x$ , by

$$C \propto \frac{r}{x^{2/3}}.$$

]

---

In this equation, we have continued to use the subscript D to denote dilute phase and C for concentrated phase.

We are given that at saturated vapour pressure, the effective mass ratios are

$$r_C = 2.8, \text{ and } r_D = 2.4.$$

In the concentrated phase, the concentration is nearly 100%. In the dilute phase has a limiting concentration of 6.6%. So

$$x_C = 1, \text{ and } x_D = 0.066.$$

Substituting these into the above equation for  $\Delta H$ , we find

$$\Delta H = 11.9TC_1.$$

We are given that at the pressure of 20 bar, the effective mass ratios change to

$$r_C = 4.6, \text{ and } r_D = 2.8.$$

Substituting these into the above equation for  $\Delta H$ , we find

$$\Delta H = 12.5TC_1.$$

---

### Answer

The cooling power is proportional to both the enthalpy change  $\Delta H$  and the flow rate  $\dot{n}_3$ :

$$\dot{Q} \propto x\Delta H.$$

The enthalpy change in turn depend on the entropies of the concentrated (C) and the dilute (D) phases:

$$\Delta H = T(S_D - S_C).$$

From the previous exercise, we know that the entropy is equal to the heat capacity for the Fermi gas, so that

$$\Delta H = T(C_D - C_C).$$

We are given that

$$C \propto \frac{r}{x^{2/3}}.$$

Let the constant of proportion be  $C_1$ . Then the enthalpy may be written as

$$\Delta H = TC_1 \left( \frac{r_D}{x_D^{2/3}} - \frac{r_C}{x_C^{2/3}} \right).$$

---

So the enthalpy change  $\Delta H$  increases by about 5%.

Since the cooling power is proportional to the enthalpy change  $\Delta H$ :

$$\dot{Q} \propto x\Delta H,$$

it would also increase by 5%.

Remark:

1. This shows that the cooling power may be increased by increasing the pressure.
2. We have assumed that the limiting concentration in the dilute phase remains the same. In fact it would increase. This increases the flow rate and gives a greater increase in the cooling power.

**Exercise 9**

The high temperature approximation for the magnetic heat capacity of a spin-1/2 salt is given by

$$C_a = Nk_B \left( \frac{\mu_B B}{k_B T} \right)^2.$$

The exact result is given by

$$C = Nk_B \left( \frac{\mu_B B}{k_B T} \right)^2 \operatorname{sech}^2 \left( \frac{\mu_B B}{k_B T} \right).$$

For a field of 1 T, at what temperature does the high temperature approximation for the magnetic heat capacity deviate by 5% from the exact results?

This

$$1 - \operatorname{sech}^2 x = 0.05$$

can be solved to give

$$x = 0.3230$$

or

$$\frac{\mu_B B}{k_B T} = 0.3230.$$

We are given that the field  $B$  is 1 T. Substituting this and the constants, we find

$$T = 2.08 \text{ K}.$$

Define

$$x = \frac{\mu_B B}{k_B T}.$$

Rewrite the given equations in terms of  $x$ . The high temperature approximation is

$$C_a = Nk_B x^2.$$

The exact result is

$$C = Nk_B x^2 \operatorname{sech}^2 x.$$

$\operatorname{sech} x$  is a decreasing function. When they differ by 5%, the difference is

$$\frac{C - C_a}{C_a} = 0.05.$$

Substituting the above equations, we get

$$1 - \operatorname{sech}^2 x = 0.05.$$

**Exercise 10**

CMN is a spin-1/2 paramagnetic salt. Calculate the cooling power of one mole of CMN if it is demagnetised from 2 T, 1 K to zero field. Calculate for the same experiment the heat of magnetisation which has to be removed if this salt is magnetised isothermally to 2 T.

[You are given that the magnetic ordering temperature of CMN is 2 mK.]



When cooled to zero field, an interaction field  $b$  remains. This is related to the magnetic ordering temperature  $T_c$  by

$$\mu_B b = k_B T_c.$$

Substituting the ordering temperature given and the constant, we find

$$b = 0.00298 \text{ T}.$$

This is the minimum field that remains after the applied field is reduced to zero. The final temperature  $T_f$  is then given by

$$T_f = \frac{T_i}{B_i} B_f$$

The initial temperature  $T_i$  is 1 K, and initial field  $B_i$  is 2 T. Taking the interaction field  $b$  as the final field  $B_f$ , we find

$$T_f = 0.00149 \text{ K}.$$

The cooling power is the heat needed to warm up the salt from this temperature. This is given by

$$Q = \int_{T_f}^{\infty} C dT$$

Avogadro constant  $N_A$  since we are given that there is one mole of the salt.

Substituting the above values for  $b$  and  $T_f$  and the constants, we find the cooling power:

$$Q = 0.0145 \text{ J}.$$

Next, we need to find the heat of magnetisation which has to be removed if this salt is magnetised isothermally to 2 T, at a temperature of 1 K. This can be calculated using the entropy:

$$Q = T \Delta S.$$

From the lectures on paramagnetic salts, we have seen that the entropy for the spin-1/2 salt is

$$S = N k_B \ln \left[ 2 \cosh \left( \frac{\mu_B B}{k_B T} \right) \right] - \frac{N \mu_B B}{T} \tanh \left( \frac{\mu_B B}{k_B T} \right).$$

where  $N$  should be  $N_A$  for one mole of the salt.

where  $C$  is the heat capacity. It is a good approximation to the upper limit to infinity, because the heat capacity falls as  $1/T^2$ . So  $Q$  would reach a limiting value at higher temperature.

Since the heat capacity  $C$  is obtained by differentiating the energy  $U$  with respect to temperature  $T$ , we have

$$Q = \int_{T_f}^{\infty} C dT = U(\infty) - U(T_f).$$

For a spin-1/2 salt, we know from the lectures on paramagnetic salts that the energy is given by

$$U = -N \mu_B B \tanh \left( \frac{\mu_B B}{k_B T} \right).$$

Substituting this, we find the cooling power:

$$Q = N_A \mu_B B_f \tanh \left( \frac{\mu_B B_f}{k_B T_f} \right).$$

where the final field  $B_f$  is the same as the interaction field  $b$  in this case. We have replaced the number of particles  $N$  by the

The entropy change refers to the change from the entropy at 1 K, 2T, to the maximum entropy of  $N_A k_B \ln 2$ .

Substituting 1 K, 2T into the entropy formula, we find

$$S = 1.972 \text{ J/mol}.$$

The entropy change  $\Delta S$  is obtained by subtracting this from the maximum entropy of  $N_A k_B \ln 2$ .

We can now find the heat of magnetisation:

$$Q = T \Delta S = T (N_A k_B \ln 2 - 1.972) = 3.791 \text{ J}.$$

**Exercise 11**

To which temperature does one have to refrigerate a solid containing paramagnetic ions with spin 1/2 and magnetic moments equal to one Bohr magneton in a final field of 3 T so that 75% of the atoms are polarised with their spin parallel to the external magnetic field.

**Answer**

We can solve this

$$\exp\left(\frac{2\mu_B B}{k_B T}\right) = 3.$$

for the temperature  $T$ . Rearranging, we get:

$$T = \frac{2\mu_B B}{k_B \ln 3}.$$

Substituting the value for  $B$  of 3 T and the other constants, we find

$$T = 3.67 \text{ K}.$$

A spin 1/2 salt would have 2 magnetic energy levels:  $-\mu_B B$  and  $+\mu_B B$ .

The Boltzmann factors for the 2 levels are respectively:

$$\exp\left(\frac{\mu_B B}{k_B T}\right) \text{ and } \exp\left(-\frac{\mu_B B}{k_B T}\right).$$

We need 75% of the atoms in the lower level, and 25% in the higher level.

So the ratio of the Boltzmann factor would be:

$$\exp\left(\frac{\mu_B B}{k_B T}\right) : \exp\left(-\frac{\mu_B B}{k_B T}\right) = 75 : 25,$$

or

$$\exp\left(\frac{2\mu_B B}{k_B T}\right) = 3.$$

**Exercises****Exercise 12**

At which nuclear spin temperature  $T_n$  would copper nuclei order magnetically, if we assume that this order occurs roughly when the nuclear magnetic interaction energy  $\mu b_i$  becomes comparable to the thermal energy  $k_B T_n$ ? The internal field created by the neighbouring nuclei  $b_i \approx 0.3 \text{ mT}$ .

[You are given that copper has a nuclear  $g$ -factor of 2.22. The nuclear magneton  $\mu_n = 5.051 \times 10^{-27} \text{ J T}^{-1}$ .]

When the nuclear magnetic interaction energy  $\mu b_i$  becomes comparable to the thermal energy  $k_B T_n$ , we have

$$k_B T_n = \mu b_i.$$

The magnetic moment is

$$\mu = g\mu_n.$$

Substituting this, we get

$$k_B T_n = g\mu_n b_i.$$

Solving for  $T_n$ ,

$$T_n = \frac{g\mu_n b_i}{k_B}.$$

Substituting the given values of  $g$  and  $b_i$ , we get

$$T_n = 3.66 \mu\text{T}.$$

The high temperature approximation for the entropy is given by

In order to use this,  $x$ , given by

$$x = \frac{g_n \mu_n B}{k_B T},$$

must be small. Substituting the given  $B$  of 8 T,  $T$  of 0.015 K, and  $g_n$  of 2.22, as well as the other constants, we find

$$x = 0.43.$$

This is not much smaller than one, so using the high temperature approximation would only give us an estimate. But it should simplify the calculation a lot.

The cooling power of the dilution refrigerator is

$$\dot{Q} = 84 n_3 T^2.$$

The flow rate  $n_3$  is rate of 1 mmole  $^3\text{He/s}$  that is given. As the copper block is cooled from a higher temperature.

### Exercise 13

How long would it take a  $^3\text{He}$ - $^4\text{He}$  dilution refrigerator with a circulation rate of 1 mmole  $^3\text{He/s}$  to precool 20 mole of copper in 8 T to 15 mK from a higher temperature? How does this compare with cooling it isothermally at 15 mK with the same dilution refrigerator?

You may use the high temperature approximation for the entropy per mole:

$$S_n = R \ln(2I + 1) - \frac{\lambda_n B^2}{2\mu_0 T_n^2}$$

where  $\lambda_n = N_A I(I + 1) \mu_0 \mu_n^2 g_n^2 / 3k_B$ , and  $g_n$  is the  $g$ -factor for the nucleus.

[You are given that copper has a nuclear spin  $I$  of 3/2 and a nuclear  $g$ -factor of 2.22.

The nuclear magneton  $\mu_n = 5.051 \times 10^{-27} \text{ J T}^{-1}$ .]

We also need to assume that the magnetic heat capacity is the main contribution to the heat capacity. We know from earlier lectures that the magnetic contribution becomes dominant at around this temperature range.

The heat absorbed from the copper is related to the entropy by

$$dQ = T dS.$$

This can be related to the cooling power as follows:

$$-\frac{dQ}{dt} = T \frac{dS}{dt}$$

A minus sign is introduced because absorbing heat from the copper causes its energy to decrease.

Substituting the above formula for the cooling power on the left, and the given formula for the entropy on the right, we get:

$$-84 n_3 T^2 = \frac{n \lambda_n B^2}{\mu_0 T^2} \frac{dT}{dt}.$$

A factor  $n$  has been added for the given number of moles, since entropy formula is for 1 mole of copper. The derivative of  $T$  appears on the right because temperature changes with time. Rearranging gives:

$$\int_0^{t_1} dt = -\frac{n\lambda_n B^2}{84\mu_0 n_3} \int_{\infty}^{T_i} \frac{dT}{T^4}.$$

We use  $T_i$  to represent the temperature of 15 mK that we want to reach, and  $t_1$  is the time taken.

We use infinity to approximate a high starting temperature. Since the integrating varies as  $1/T^4$ , it would become very small as high  $T$ , so this should be all right.

Integrating, we find the time taken:

$$t_1 = \frac{1}{3} \frac{n\lambda_n B^2}{84\mu_0 n_3 T_i^3}.$$

Substituting the values:

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remains fixed. The heat of magnetisation can be calculated using  $TdS$ . Since temperature is constant, the heat change is

$$\Delta Q = T\Delta S.$$

The entropy change can be calculate from the the high temperature approximation for the entropy per mole:

$$S_n = R\ln(2I + 1) - \frac{\lambda_n B^2}{2\mu_0 T_n^2}.$$

At zero field ( $B = 0$  T),

$$S_n = R\ln(2I + 1).$$

So the entropy change is obtained by subtracting the 2 equations above:

$$\Delta S = n \frac{\lambda_n B^2}{2\mu_0 T_i^2}.$$

where  $n$  is the number of moles of copper. The heat change is

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$$\begin{aligned} n &= 20, \\ B &= 8 \text{ T}, \\ T_i &= 0.015 \text{ K and} \\ n_3 &= 0.001 \text{ mole/s,} \end{aligned}$$

we find the time taken:

$$t_1 = 10300 \text{ s.}$$

This is 2 hours and 52 minutes - quite fast in low temperature cooling.

Next, we need to compare this with isothermal cooling. Instead of starting at a higher temperature and cool down at a fixed field, we start at the low temperature and increase the field.

This is much easier to calculate. Since the temperature is constant the cooling power

$$\dot{Q} = 84n_3 T_i^2$$

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then

$$\Delta Q = T\Delta S = \frac{n\lambda_n B^2}{2\mu_0 T_i}.$$

Let the time taken for the magnetisation be  $t_2$ . This time would depend on how fast the heat produced can be removed, i.e. the cooling rate  $\dot{Q}$  of the dilution refrigerator. So the time taken is:

$$t_2 = \frac{\Delta Q}{\dot{Q}}.$$

Substituting the above equations, we find:

$$t_2 = \frac{1}{2} \frac{n\lambda_n B^2}{84\mu_0 n_3 T_i^3}.$$

Comparing with the formula for  $t_1$  above, we find that this time is longer, by 50%.

So it is faster in this case to cool down from a higher temperature!

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